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A class of boundary traces for solutions of the equation $Lu = \psi(u)$

E.B. Dynkin [a,*,1], S.E. Kuznetsov [b]

[a] *Department of Mathematics, Cornell University, Ithaca, NY 14853, USA*
[b] *Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, USA*

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Abstract

Our subject is the class $\mathcal{U}$ of all positive solutions of a semilinear equation $Lu = \psi(u)$ in E where L is a second order elliptic differential operator, E is a domain in $\mathbb{R}^d$ and ψ belongs to a convex class Ψ of C^1 functions which contains functions $\psi(u) = u^\alpha$ with $\alpha > 1$. A special role is played by a class $\mathcal{U}_0$ of solutions which we call σ-moderate. A solution u is moderate if there exists $h \geqslant u$ such that $Lh = 0$ in E. We say that $u \in \mathcal{U}$ is σ-moderate if u is the limit of an increasing sequence of moderate solutions. In [E.B. Dynkin, S.E. Kuznetsov, Fine topology and fine trace on the boundary associated with a class of quasilinear differential equations, Comm. Pure Appl. Math. 51 (1998) 897–936] all σ-moderate solutions were classified by using their fine boundary traces.[2] In [M. Marcus, L. Véron, The precise boundary trace of positive solutions of the equation $\Delta u = u^q$ in the supercritical case, in: Perspectives in Nonlinear Partial Differential Equations, in honor of Haim Brezis, Contemp. Math., Amer. Math. Soc., Providence, RI, 2007, arxiv.org/math/0610102] Marcus and Véron introduced a different characteristic (called the precise trace) for solutions of the equation $\Delta u = u^\alpha$ with $\alpha > 1$ in a bounded C^2 domain. In the present paper we develop a general scheme covering both approaches and we prove the equivalence, in a certain sense, of the fine and precise traces. An implication of this equivalence is a Wiener type criterion for vanishing of the Poisson kernel of the equation $Lu(x) = a(x)u(x)$.

* Corresponding author.
E-mail addresses: ebd1@cornell.edu (E.B. Dynkin), sek@euclid.colorado.edu (S.E. Kuznetsov).

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2 It is known (see [B. Mselati, Classification and probabilistic representation of the positive solutions of a semilinear elliptic equation, Mem. Amer. Math. Soc. 168 (798) (2004); E.B. Dynkin, Superdiffusions and Positive Solutions of Nonlinear Partial Differential Equations, Amer. Math. Soc., Providence, RI, 2004]) that $\mathcal{U}_0 = \mathcal{U}$ in the case of the equation $\Delta u = u^\alpha$ with $1 < \alpha \leqslant 2$ in a bounded smooth domain. Therefore in this case the description of all solutions of $Lu = \psi(u)$ follows from the results on σ-moderate solutions.

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1. Introduction

1.1. Program

We consider a second order differential operator

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} \tag{1.1}$$

in a bounded C^2 domain $E \subset \mathbb{R}^d$. Every positive solution of equation

$$Lh = 0 \quad \text{in } E \tag{1.2}$$

has a unique representation

$$h(x) = \int_{\partial E} k(x,y)\nu(dy), \tag{1.3}$$

where $k(x,y)$ is the Poisson kernel for L in E and $\nu \in \mathcal{M}(\partial E)$.[3] Formula (1.3) establishes a 1–1 mapping from $\mathcal{M}(\partial E)$ onto the set of all positive solutions of (1.2).[4]

With every solution of the equation

$$Lu = \psi(u) \quad \text{in } E \tag{1.4}$$

we associate a subset $S(u)$ of ∂E (called the singular set for u) and a measure ν_u on $\partial E \setminus S(u)$. We use notation $\operatorname{tr}(u)$ for the pair $(S(u), \nu_u)$ and we call this pair the boundary trace of u. We introduce a class A of traces. Both fine trace of Dynkin and Kuznetsov (DK-trace) and precise trace of Marcus and Véron (MV-trace) belong to this class. All traces of this class are equivalent in a certain sense.

1.2. Tools

To investigate the class $\mathcal{U}$ of solutions of (1.4) we work with the class $\mathcal{U}_1$ of moderate solutions and the class $\mathcal{U}_0$ of σ-moderate solutions defined in the Abstract and we use the following tools.[5]

[3] $\mathcal{M}(A)$ means the set of all finite measures on the space A.

[4] For an arbitrary domain $E \subset \mathbb{R}^d$, the Poisson kernel and ∂E should be replaced by the Martin kernel and a certain Borel subset E' of the Martin boundary. The condition that E is a bounded C^2 domain is used only in Section 4 but to keep the same notation through the entire article we write everywhere ∂E, not E'.

[5] See Section 2 for a more complete description of these tools.

(1.2.A) A lattice structure in $\mathcal{U}$ determined by the partial order $u \leqslant v$. For every $\tilde{\mathcal{U}} \subset \mathcal{U}$ there exists the supremum $\operatorname{Sup} \tilde{\mathcal{U}}$ relative to this partial order. We put

$$\begin{aligned} u \oplus v &= \operatorname{Sup}\{w \in \mathcal{U}:\ w \leqslant u + v\}, \\ u \vee v &= \operatorname{Sup}\{u, v\}. \end{aligned} \tag{1.5}$$

If $\tilde{\mathcal{U}}$ is closed under $\vee$, then there exists an increasing sequence $u_n \in \tilde{\mathcal{U}}$ converging pointwise to $\operatorname{Sup} \tilde{\mathcal{U}}$.

(1.2.B) A monotone 1–1 mapping $\nu \to u_\nu$ from a subset $\mathcal{N}_1$ of $\mathcal{M}(\partial E)$ onto $\mathcal{U}_1$ which we continue to a monotone mapping from a set $\mathcal{N}_0$ onto $\mathcal{U}_0$ such that $u_\mu \oplus u_\nu = u_{\mu+\nu}$ for all $\mu, \nu \in \mathcal{N}_0$.

(1.2.C) A monotone mapping $B \to u_B$ from the Borel σ-algebra $\mathcal{B} = \mathcal{B}(\partial E)$ to $\mathcal{U}_0$ defined by the formula

$$u_B = \operatorname{Sup}\{u_\nu:\ \nu \in \mathcal{N}_1,\ \nu \text{ is concentrated on } B\}. \tag{1.6}$$

(1.2.D) The class $\mathcal{P}$ of sets which do not charge any $\nu \in \mathcal{N}_1$. We call them *polar sets*.

We write $B_1 \sim B_2$ if the symmetric difference of B_1 and B_2 is polar. For every class $\mathcal{A}$ of sets in ∂E we denote by $\bar{\mathcal{A}}$ the class of all sets A' equivalent to sets $A \in \mathcal{A}$. By (1.6), $u_{B_1} = u_{B_2}$ if $B_1 \sim B_2$. Writing $\nu_1 \sim \nu_2$ means that measures ν_1 and ν_2 coincide off a polar set. If $\nu_1, \nu_2 \in \mathcal{N}_0$, this implies $u_{\nu_1} = u_{\nu_2}$. We write $(\Gamma, \nu) \sim (\Gamma', \nu')$ if $\Gamma \sim \Gamma'$ and $\nu \sim \nu'$.

1.3. A-traces

We consider mapping $\beta : \bar{\mathcal{B}} \to \bar{\mathcal{B}}$ subject to the conditions:

(1.3.A) $\beta(\Gamma) \in \mathcal{P}$ if $\Gamma \in \mathcal{P}$.
(1.3.B) $\beta(B \cup \Gamma) = \beta(B) \cup \beta(\Gamma)$.
(1.3.C) For all $\Gamma \in \mathcal{B}$, $\beta(\Gamma) \cap \Gamma \sim \Gamma$.

By (1.3.B) the class

$$\mathcal{T} = \big\{\Gamma:\ \beta(\Gamma) \subset \Gamma\big\} \tag{1.7}$$

is closed under finite unions. Put $B \in \mathbb{F}$ if B is the intersection of a collection of sets $\tilde{\mathcal{T}} \subset \mathcal{T}$. Since $\mathbb{F}$ is closed under intersections and finite unions, it is the class of all closed sets in a topology. We call it $\mathcal{T}$*-topology*. Let $\mathbb{O}$ stands for the class of all open sets in this topology.

Consider a mapping β with properties (1.3.A)–(1.3.C) and a mapping S from $\mathcal{U}$ to $\bar{\mathcal{B}}$ such that

(1.3.D) $\beta(\Gamma) \sim S(u_\Gamma)$.

We say that a map $u \to (S(u), \nu_u)$ determines the *boundary trace of class* A if:

(1.3.E) For every $u \in \mathcal{U}_0$, $\Gamma = S(u)$ and $\nu = \nu_u$ have the following properties:
(1.3.E1) $\Gamma \in \bar{\mathbb{F}}$.

(1.3.E2) $\nu \sim \tilde{\nu}$ where $\tilde{\nu}$ is a σ-finite measure of class $\mathcal{N}_0$ such that $\tilde{\nu}(\Gamma) = 0$.
(1.3.E3) $S(u_\nu) \subset \Gamma'$ where $\Gamma' \sim \Gamma$.

(1.3.F) If (Γ, ν) satisfy conditions (1.3.E1)–(1.3.E3), then $u = u_\Gamma \oplus u_\nu$ is a unique σ-moderate solution such that $\mathrm{tr}(u) \sim (\Gamma, \nu)$.

(1.3.G) If $u, v \in \mathcal{U}_0$ and if $S(u) \sim S(v)$, $\nu_u \sim \nu_v$, then $u = v$.

Proposition 1.1. *An existence of an A-trace implies*: *every σ-moderate solution has a representation*

$$u = u_\Gamma \oplus u_\nu, \tag{1.8}$$

where (Γ, ν) *satisfy conditions* (1.3.E1)–(1.3.E3).

Proof. By (1.3.E), conditions (1.3.E1)–(1.3.E3) hold for $(\Gamma, \nu) = \mathrm{tr}(u)$. By (1.3.F), $u_\Gamma \oplus u_\nu$ is a unique σ-moderate solution with the trace equivalent to (Γ, ν) which implies (1.8). □

1.4. Equivalent traces

We say that two traces tr and tr′ are *equivalent* if $\mathrm{tr}(u) \sim \mathrm{tr}'(u)$ for all σ-moderate u. Then we have:

(1.4.A) All traces of class A are equivalent.

Indeed, if $u \in \mathcal{U}_0$, then, by Proposition 1.1, $u = u_\Gamma \oplus u_\nu$ with (Γ, ν) subject to conditions (1.3.E1)–(1.3.E3). If tr and tr′ are A-traces, then, by (1.3.F), both $\mathrm{tr}(u)$ and $\mathrm{tr}'(u)$ are equivalent to (Γ, ν).

Clearly, we also have:

(1.4.B) Class A contains with every trace all equivalent traces.

We verify in Section 3 that DK-trace is an A-trace. In Section 4 we deduce from the results in [9] that MV-trace is also an A-trace. Hence both traces are equivalent.

1.5. Assumptions about L and ψ

We assume that the coefficients of the operator L defined by (1.1) satisfies the conditions:

(1.5.A) (Uniform ellipticity.) There exists a constant $\kappa > 0$ such that

$$\sum a_{ij}(x) t_i t_j \geqslant \kappa \sum t_i^2 \quad \text{for all } x \in E, t_1, \dots, t_d \in \mathbb{R}.$$

(1.5.B) $a_{ij}(x)$ and $b_i(x)$ are bounded and Hölder continuous.

We assume that ψ is a function on $\mathbb{R}_+ = [0, \infty)$ with the properties:

(1.5.C) ψ is a convex function of class $C^1(\mathbb{R}_+)$.
(1.5.D) $\psi(0) = \psi'(0) = 0$.

(1.5.E) There is a constant a such that

$$\psi(2u) \leqslant a\psi(u)$$

for all u.[6]

(1.5.F) $\int_N^\infty ds[\int_0^s \psi(u)\,du]^{-1/2} < \infty$ for some $N > 0$.

Keller [6] and Osserman [10] proved independently: the condition (1.5.F) implies that functions $u \in \mathcal{U}(E)$ are uniformly bounded on every compact subset of E.[7]

Conditions (1.5.C)–(1.5.F) hold for $\psi(u) = u^\alpha$ with $\alpha > 1$.

2. More on lattice structure in $\mathcal{U}$ and on labeling of σ-moderate solutions

2.1. Lattice structure in $\mathcal{U}$

For every $\tilde{\mathcal{U}} \subset \mathcal{U}$, there exists a unique element u of $\mathcal{U}$ with the properties:

(a) $u \geqslant v$ for every $v \in \tilde{\mathcal{U}}$;
(b) if $\tilde{u} \in \mathcal{U}$ satisfies (a), then $u \leqslant \tilde{u}$.

We denote this element $\operatorname{Sup} \tilde{\mathcal{U}}$.

For every $u, v \in \mathcal{U}$, we define $u \oplus v$ and $u \vee v$ by formula (1.5).

In general, $\operatorname{Sup} \tilde{\mathcal{U}}$ does not coincide with the pointwise supremum (the latter does not belong to $\mathcal{U}$). However, both are equal if $u \vee v \in \tilde{\mathcal{U}}$ for all $u, v \in \tilde{\mathcal{U}}$. Moreover, in this case there exist $u_n \in \tilde{\mathcal{U}}$ such that $u_n(x) \uparrow u(x)$ for all $x \in E$. Therefore, if $\tilde{\mathcal{U}}$ is closed under $\vee$ and if it consists of moderate solutions, then $\operatorname{Sup} \tilde{\mathcal{U}}$ is σ-moderate.

The classes $\mathcal{U}_1$ and $\mathcal{U}_0$ are closed under $\vee$ and $\oplus$.

2.2. Labeling of σ-moderate solutions

We introduce a class $\mathcal{N}_0$ of measures on ∂E and a mapping $\nu \to u_\nu$ from $\mathcal{N}_0$ to $\mathcal{U}_0$ with the following properties.

(2.A) $u_\mu \leqslant u_\nu$ for $\mu \leqslant \nu$.
(2.B) $u_{\nu_n} \uparrow u_\nu$ as $\nu_n \uparrow \nu$.

This is done in three steps.

Step 1. We denote h_ν the function corresponding to a measure $\nu \in \mathcal{M}(\partial E)$ by formula (1.3).

Step 2. *Labeling of moderate solutions u.* We put $\nu \in \mathcal{N}_1$ and $u = u_\nu$ if h_ν is the minimal solution of (1.2) bigger than u.

[6] In the literature this is called Δ_2 condition.
[7] A proof in a more general setting can be found in [2, Chapter 5, Section 3].

Step 3. *Labeling of σ-moderate solutions u.* We put $\nu \in \mathcal{N}_0$ if there exists a sequence of $\nu_n \in \mathcal{N}_1$ such that $\nu_n \uparrow \nu$, and we put $u_\nu = \lim u_{\nu_n}$.[8]

We prove that this labeling satisfies conditions (2.A), (2.B).[9] Moreover,

$$u_\mu \vee u_\nu = u_{\mu \vee \nu}, \qquad u_\mu \oplus u_\nu = u_{\mu+\nu}. \tag{2.1}$$

(A label ν is defined uniquely for a moderate solution but, in general, the equality $u_{\nu_1} = u_{\nu_2}$ does not imply that $\nu_1 = \nu_2$.)

3. DK-trace

In this section we verify that the fine trace described in [2,4] is an A-trace.
According to the definition[10] $\mathrm{tr}(u) = (\Gamma, \nu)$ where

$$\begin{gathered} \Gamma = S(u), \\ \nu(B) = \sup\bigl\{\mu(B)\colon\ \mu \in \mathcal{N}_1,\ \mu(\Gamma) = 0,\ u_\mu \leqslant u\bigr\}. \end{gathered} \tag{3.1}$$

We describe $S(u)$ in the next section.
The topology $\mathcal{T}$ corresponds to

$$\beta(\Gamma) = S(u_\Gamma)$$

which implies (1.3.D).

3.1. Singular set $S(u)$

Heuristically, $y \in S(u)$ if y is a point of rapid growth of $\psi'(u)$. (A special role of $\psi'(u)$ is due to the fact that the tangent space to $\mathcal{U}$ at point u is described by the equation $Lv = \psi'(u)v$.)
The rapid growth of a positive continuous function $a(x)$ can be defined analytically or probabilistically.

Probabilistic Definition. It involves the conditioning (ξ_t, Π_x^y) of the diffusion $\xi = (\xi_t, \Pi_x)$ given that the ξ hits ∂E at point y.[11]
A rate of growth at $y \in \partial E$ of a positive continuous function $a(x)$ can be characterized by the function

$$v(x) = \Pi_x^y \exp\left\{-\int a(\xi_t)\, dt\right\} \tag{3.2}$$

[8] The limit is the same for all sequences $\nu_n \uparrow \nu$.
[9] See [2, p. 120].
[10] See [2, Chapter 11, formula (7.1)].
[11] The measure Π_x^y is constructed as h-transform of Π_x with $h(x) = k(x, y)$ where $k(x, y)$ is the Poisson (or Martin) kernel. This construction, due to Doob, is described, for instance, in [2, Chapter 7, Section 3].

(the integral over the entire path till hitting ∂E). We call y a point of rapid growth of $a(x)$ if $v(x) = 0$ for all x which is equivalent to the condition

$$\int a(\xi_t)\,dt = \infty, \quad \Pi_x^y\text{-a.s. for all } x \in E. \tag{3.3}$$

Thus

$$S(u) = \left\{ y \in \partial E\colon \int \psi'\bigl[u(\xi_t)\bigr]\,dt = \infty,\ \Pi_x^y\text{-a.s. for all } x \in E \right\}. \tag{3.4}$$

Remark. If $a = \psi'(u)$, then (3.5) is the linearization of Eq. (1.4). $Lv - av$ is the generator of the diffusion (ξ_t, Π_x) killed at x with the rate $a(x)$. As a result of the killing, a path does not hit, a.s., the set of points where a grows rapidly.

Analytic Definition. Function v is a solution of a linear equation

$$Lv - av = 0 \tag{3.5}$$

(see [2, Chapter 7, Section 3]). Moreover formula (3.2) defines the Poisson kernel $k_a(x, y)$ for the operator $Lv - av$. Hence,

$$S(u) = \bigl\{ y \in \partial E\colon k_{\psi'(u)}(x, y) = 0 \text{ for all } x \in E \bigr\}. \tag{3.6}$$

3.2. Fine trace belongs to class A

Conditions (1.3.A)–(1.3.G) follow from [2, Chapter 11]. In particular, (1.3.C) follows from (4.2.D), (1.3.E) is an implication of Theorem 7.1.A and (1.3.F) is implied by Theorem 7.1.B.

It remains to verify (1.3.G). Suppose $\mathrm{tr}(u) = (\Gamma, \nu)$, $\mathrm{tr}(v) = (\Gamma', \nu')$. If $S(u) \sim S(v)$, then $\Gamma \sim \Gamma'$ and $u_\Gamma = u_{\Gamma'}$. Since $\nu = \nu'$, we have $u_\Gamma \oplus u_\nu = u_{\Gamma'} \oplus u_{\nu'}$. It follows from the concluding part of Theorem 7.1.B that, if $u, v \in \mathcal{U}_0$, then $u = u_\Gamma \oplus u_\nu$, $v = u_{\Gamma'} \oplus u_{\nu'}$ and therefore $u = v$.

4. MV-trace

It is defined for the solutions of $\Delta u = u^\alpha$ with $\alpha > 1$ in a bounded C^2 domain E. In this case ∂E is a C^2 manifold and it is possible to use the Bessel capacities $C_{\ell,p}$ on ∂E. The capacity $\mathrm{Cap}_\alpha = C_{2/\alpha,\alpha'}$ is intimately related to the equation $\Delta u = u^\alpha$.[12] In particular, $\Gamma \subset \partial E$ is polar if and only if $\mathrm{Cap}_\alpha(\Gamma) = 0$.[13]

[12] Instead of Cap_α it is possible to consider an equivalent Poisson capacity. (See [5].)
[13] See [2, Theorem 1.1 in Chapter 12 and Theorem 0.1 in Chapter 13].

4.1. Singular set $S(u)$ and mapping $\beta(\Gamma)$

The paper [9] is devoted to supercritical case $\alpha \geqslant (d+1)/(d-1)$.[14] $\beta(\Gamma)$ is defined as the set of $y \in \partial E$ at which Γ is thick. This is formulated in terms of Cap_α. For every $y \in \partial E$, denote by Γ_y^r the intersection of Γ with the ball in $\mathbb{R}^d$ of radius r centered at y and put

$$\lambda_y^r(\Gamma) = \mathrm{Cap}_\alpha\bigl(\Gamma_y^r\bigr),$$
$$\Lambda_y(\Gamma) = \int_0^1 \left(\frac{\lambda_y^r(\Gamma)}{r^{d-\ell p}}\right)^{p'-1} \frac{dr}{r}. \tag{4.1}$$

(Recall that we consider the case $\ell = 2/\alpha$, $p = \alpha'$.)[15] A set Γ is thick at y if $\Lambda_y(\Gamma) = \infty$. If Γ is polar, then $\mathrm{Cap}_\alpha(\Gamma) = 0$ and therefore β satisfies (1.3.A). It satisfies also (1.3.B). Indeed, if $\gamma > 0$, then there exists a constant c such that $(s+t)^\gamma \leqslant c(s^\gamma + t^\gamma)$ for all $s, t \geqslant 0$. Therefore

$$\Lambda_y(B) \vee \Lambda_y(\Gamma) \leqslant \Lambda_y(B \cup \Gamma) \leqslant c\bigl(\Lambda_y(B) + \Lambda_y(\Gamma)\bigr)$$

which implies (1.3.B). Property (1.3.C) is proved in [1, p. 175]. (A standard reference regarding topologies associated with Bessel capacities is [1, Chapter 6].)

The definition of $S(u)$ is based on the following property of a bounded C^2 domain E. Denote $\Sigma(\beta, y)$ the point z on the inner normal to ∂E at y at the distance β from y. For sufficiently small $\varepsilon > 0$ the mapping Σ is a C^2 diffeomorphism from $(0,\varepsilon) \times \partial E$ onto $E_\varepsilon = \{x \in E\colon\ \rho(x) < \varepsilon\}$ where $\rho(x) = \mathrm{dist}(x, \partial E)$. Denote by D_O the image of $(0,\varepsilon) \times O$ under this diffeomorphism and put

$$V_O(u) = \int_{D_O} u(x)^\alpha \rho(x)\, dx.$$

$S(u)$ can be defined as the set of $y \in \partial E$ such that $V_O(u) = \infty$ for all $\mathcal{T}$-neighborhoods of y. It follows from inequalities $u \vee v \leqslant u \oplus v \leqslant u + v$ that

$$S(u \oplus v) = S(u) \cup S(v).$$

According to [9, Theorem 1.1], $S(u)$ can also be defined as follows. For every $O \in \mathbb{O}$ and every $0 < \beta < \varepsilon$ consider the image $\Sigma_\beta(O)$ of $\{\beta\} \times O$ under the diffeomorphism Σ and the surface area γ_β on this image. We associate with every solution $u \in \mathcal{U}$ two classes of sets $O \in \mathbb{O}$. For both classes there exists a limit

$$\lim_{\beta \to 0} \int_{\Sigma_\beta(O)} u\, d\gamma_\beta.$$

[14] The subcritical case $\alpha < (d+1)/(d-1)$ is covered in [7].

[15] In this case $0 < p \leqslant d/\ell$ for supercritical α—a condition needed for a number of results used in [9].

Put $O \in \mathbb{O}'$ if this limit is finite and $O \in \mathbb{O}''$ if it is infinite. By [9, Theorem 1.1], $y \in S(u)$ if all $\mathcal{T}$-neighborhoods of y belong to $\mathbb{O}''$, and y belongs to the complement $R(u)$ of $S(u)$ if a $\mathcal{T}$-neighborhood of y belongs to $\mathbb{O}'$. Hence $R(u) \in \mathbb{O}$ and $S(u) \in \mathbb{F}$.

We consider several versions of the trace.

4.2. *Traces* $\mathrm{tr}^c(u)$ *and* $\mathrm{tr}^d(u)$

The trace $\mathrm{tr}^c(u)$ is a pair $(S(u), \nu_u)$. Measure ν_u is defined by an operation $u \to Q_O(u)$ which is natural to call the sweeping of u to O. Put $E_{>\beta} = \{x \in E\colon \operatorname{dist}(x, \partial E) > \beta\}$. Denote by u_β^O the solution of the integral equation

$$v(x) = \int\limits_{E_{>\beta}} g_\beta(x, y) v^\alpha(y)\, dy + \int\limits_{\Sigma_{\beta(O)}} k^\beta(x, y) u(y) \gamma_\beta(dy),$$

where g^β is the Green function and k^β is the Poisson kernel of Δ in $E_{>\beta}$.[16]

By [9, Theorem 1.2], *for every* $y \in R(u)$ *there exists a* $\mathcal{T}$*-neighborhood* O *such that*: (a) $\mathcal{T}$*-closure of* O *is contained in* $R(u)$; (b) u_β^O *converge locally uniformly as* $\beta \to 0$ *to a moderate solution* u_{ν_O} *which is denoted* $Q_O(u)$. *There exists an outer* $\mathcal{T}$*-regular measure* ν_u *on* ∂E *concentrated on* $R(u)$ *such that* $\nu_u = \nu_O$ *on* O *for every* O.[17]

It follows from [9, Theorem 5.16] that $(\Gamma, \nu) = \mathrm{tr}^c(u)$ for some $u \in \mathcal{U}$ if and only if (Γ, ν) belongs to the class $\mathfrak{C}$ determined by the conditions:

(4.2.A) $\Gamma \in \mathbb{F}$, $\nu(\Gamma) = 0$.
(4.2.B) If $O \in \mathbb{O}$ and $\nu(O) < \infty$, then $\nu(B) = 0$ for all polar Borel sets $B \subset O$.
(4.2.C) Measure ν is locally finite, i.e., every point $y \in V = \partial E \setminus \Gamma$ has a $\mathcal{T}$-neighborhood of finite measure ν.
(4.2.D) $\Gamma = \beta(\Gamma) \cup Ex(\nu)$ where $Ex(\nu) = \{y\colon \nu(O) = \infty$ for every $\mathcal{T}$-neighborhood O of $y\}$ is the set of explosion points of ν.
(4.2.E) ν is outer $\mathcal{T}$-regular.

It follows from (4.2.C) and the quasi-Lindelöf property,[18] that:

(4.2.F) ν is σ-finite on a $\mathcal{T}$-open subset V^d of V such that $N = V \setminus V^d$ is polar.

With every pair $(\Gamma, \nu) \in \mathfrak{C}$ we associate a pair (Γ^d, ν^d) defined by the formulae:

$$\Gamma^d = \Gamma \cup N,$$
$$\nu^d = \begin{cases} \nu\nu_u & \text{on } V^d, \\ 0 & \text{on } \Gamma^d. \end{cases}$$

[16] This is equivalent to the definition of u_O^β in [9] as a solution of the equation $\Delta v = v^\alpha$ in $E_{>\beta}$ subject to the boundary conditions $v = u$ on $\Sigma_\beta(O)$, $v = 0$ on $\partial E_{>\beta} \setminus \Sigma_\beta(O)$.

[17] A measure ν is called outer $\mathcal{T}$-regular if, for every Borel set B, $\nu(B)$ is equal to $\inf \nu(O)$ over all $O \in \mathbb{O}$ such that $O \supset B$.

[18] See [1, p. 184].

We introduce a modification $\mathrm{tr}^d(u)$ of $\mathrm{tr}^c(u)$ by setting $\mathrm{tr}^d(u) = (\Gamma^d, \nu^d)$ if $S(u) = \Gamma$, $\nu_u = \nu$. Clearly, if $(\Gamma, \nu) \in \mathfrak{C}$, then (Γ^d, ν^d) has the properties:

(4.2.G) $\Gamma^d \in \bar{\mathbb{F}}$.
(4.2.H) ν^d is a σ-finite measure of class $\mathcal{N}_0$.

The following property will be established in Section 4.3.

(4.2.I) $S(u_{\nu^d}) \subset \Gamma$.

4.3. *Trace* $\mathrm{tr}^a(u)$

To every pair $(\Gamma, \nu) \in \mathfrak{C}$ there corresponds a measure on ∂E defined by the formula

$$\mathbb{T}(\Gamma, \nu) = \begin{cases} \nu & \text{on } \partial E \setminus \Gamma, \\ \infty & \text{on } \Gamma. \end{cases} \tag{4.2}$$

By [9, Theorem 5.16] $\mathbb{T}$ is a 1–1 mapping from $\mathfrak{C}$ onto the class of measures μ characterized by the conditions:

(4.3.A) For every $O \in \mathbb{O}$ and for all polar $B \in \mathcal{B}$, $\mu(O) = \mu(O \cap B^c)$.
(4.3.B) μ is outer regular in topology $\mathcal{T}$.

Following [9], we call measures with properties (4.3.A), (4.3.B) perfect and we denote this class by $\mathbb{M}$. By definition, $\mathrm{tr}^a(u) = \mu$ if $\mathrm{tr}^c(u) = (S, \nu)$ and if $\mu = \mathbb{T}(S, \nu)$.

By [9, Theorem 1.3], $\mu = \mathrm{tr}^a(u)$ for some $u \in \mathcal{U}$ if and only if μ is perfect which implies:

(4.3.C) A solution $u \in \mathcal{U}$ such that

$$\mathrm{tr}^c(u) = (\Gamma, \nu) \tag{4.3}$$

exists if and only if $(\Gamma, \nu) \in \mathfrak{C}$. Moreover (4.3) holds for a σ-moderate solution

$$u = u_\Gamma \oplus u_\nu, \tag{4.4}$$

where u_Γ is defined by (1.6).[19]

Property (4.2.I) follows from (4.3.C). Indeed, by (4.3.C), $\mathrm{tr}^c(u_\Gamma \oplus u_\nu) = (\Gamma, \nu)$ and therefore $S(u_\Gamma \oplus u_\nu) = \Gamma$ and $S(u_\nu) \subset \Gamma$. Since $\nu^d \sim \nu$, we conclude that $u_{\nu^d} = u_\nu$.

The following result is a part of [9, Theorem 1.4].

[19] In Theorem 1.3 [9] formula $u = U_S \oplus v$ stands for (4.4). However $U_S = u_S$ according to [3,8]. Function v is defined as the supremum of all moderate u_τ corresponding to restrictions τ of μ to sets $O \in \mathbb{O}$ of finite measure μ. It is easy to see that $v = u_\nu$.

(4.3.D) If $u, v \in \mathcal{U}_0$ and if $\mathrm{tr}^c(u) = \mathrm{tr}^c(v)$, then $u = v$. The solution (4.4) is a unique σ-moderate solution such that $\mathrm{tr}^c(u) = (S, \nu)$.[20]

4.4. MV-trace belongs to class A

Theorem 4.1. tr^c *and* tr^d *are equivalent and they belong to class A.*

Proof. It is sufficient to show that tr^d is an A-trace and that it is equivalent to tr^c. The equivalence of tr^d and tr^c follows immediately from the definition of tr^d. Properties (1.3.E), (1.3.F) and (1.3.G) of tr^d are implications of (4.2.G)–(4.2.I) and (4.3.C), (4.3.D). Finally, (1.3.D) follows from Theorem 4.2 which will be proved in the next section. □

4.5. Relation between S and β

Theorem 4.2. *For every* $\Gamma \in \mathcal{B}$,

$$\beta(\Gamma) = S(u_\Gamma).$$

Denote by $\tilde{B}$ the $\mathcal{T}$-closure of B. First we prove several lemmas.

Lemma 4.1. $F = \beta(\Gamma) \in \mathbb{F}$ *for every* $\Gamma \in \mathcal{B}$. *Moreover,* $\beta(F) = F$.

Proof. To prove that $\beta(F) \subset \beta(\Gamma)$ we use a family $\mathcal{V}$ of positive lower semicontinuous functions with the properties:

(a) If $a \in \tilde{B}$ is not in $\beta(B)$, then $v(a) < v^*_B(a)$ for some $v \in \mathcal{V}$ where

$$v^*_B(a) = \liminf_{x \to a,\, x \in B \setminus \{a\}} v(x).$$

(b) If $a \in \beta(B)$, then $v(a) = v^*_B(a)$ for all $v \in \mathcal{V}$.

The existence of such a family follows from [1, Theorem 6.3.11].[21] We conclude that $a \in \tilde{B}$ belongs to $\beta(B)$ if and only if $v(a) = v^*_B(a)$ for all $v \in \mathcal{V}$.

Fix $a \in \beta(F)$. Since $\beta(F) \subset \tilde{F}$, $v(a) = v^*_F(a)$ for all $v \in \mathcal{V}$. There exists $a_n \in F$ such that

$$a_n \to a, \qquad v(a_n) \to v^*_F(a) = v(a).$$

Since $a_n \in \beta(\Gamma)$, we have $v^*_\Gamma(a_n) = v(a_n)$. Hence there exist $x_n \in \Gamma$ such that $x_n \to a_n$ and

$$v(x_n) \geqslant v^*_\Gamma(a_n) - 1/n = v(a_n) - 1/n \to v(a) \quad \text{as } n \to \infty.$$

Since v is lower semicontinuous, this implies $v^*_\Gamma(a) = v(a)$. Hence $a \in \beta(\Gamma)$.

20 A question of the existence of non-σ-moderate solutions remains open in the case $\alpha > 2$. However, by Theorem 1.4 [9], (4.4) is the maximal among all solutions with the trace $\mathrm{tr}^c(u) = (S, \nu)$.

21 In notation of [1], $\mathcal{V}$ consists of functions $\mathcal{V}^\mu_{2/\alpha,\alpha'}$ where μ is the capacitary measure for B corresponding to Cap_α.

To prove the second part of the lemma we note that, by (1.3.C), $F \cap \Gamma \sim \Gamma$ and therefore, by (1.3.A), (1.3.B), $F = \beta(\Gamma) \sim \beta(F \cap \Gamma) \subset \beta(F)$. However, $\beta(F) \subset F$ by the first part. $\square$

Denote by $\mathcal{N}_1(\Gamma)$ the set of all $\nu \in \mathcal{N}_1$ concentrated on Γ. For every $u \in \mathcal{U}$ denote by $\mathcal{B}(u)$ the class of all Borel sets Γ such that $u_\nu \leqslant u$ for all $\nu \in \mathcal{N}_1(\Gamma)$ and $u_\nu \nleqslant u$ for all $\nu \in \mathcal{N}_1 \setminus \mathcal{N}_1(\Gamma)$.

Lemma 4.2. *If B and Γ belong to $\mathcal{B}(u)$, then $B \sim \Gamma$.*

Proof. An assumption that $B \nsim \Gamma$ leads to a contradiction. Indeed, if $B \nsim \Gamma$, then there exists $\nu \neq 0$ of class $\mathcal{N}_1(\Gamma \cap B^c)$ or of class $\mathcal{N}_1(\Gamma^c \cap B)$. If $\nu \in \mathcal{N}_1(\Gamma \cap B^c)$, then $u_\nu \leqslant u$ (because $\nu \in \mathcal{N}_1(\Gamma)$) and $u_\nu \nleqslant u$ (because $\nu \notin \mathcal{N}_1(B)$). The case $\nu \in \Gamma^c \cap F$ can be treated in a similar way. $\square$

Lemma 4.3. *Put*

$$\mathcal{L}_1(\Gamma) = \{\nu \in \mathcal{N}_1\colon\ u_\nu \leqslant u_\Gamma\}$$

and

$$\mathcal{L}_2(\Gamma) = \{\nu \in \mathcal{N}_1\colon\ u_\Gamma \oplus u_\nu = u_\Gamma\}.$$

For every $\Gamma \in \mathcal{B}$,

$$\mathcal{N}_1(\Gamma) \subset \mathcal{L}_1(\Gamma) \subset \mathcal{L}_2(\Gamma) \tag{4.5}$$

and if $\Gamma \in \mathbb{F}$, then

$$\mathcal{N}_1(\Gamma) = \mathcal{L}_1(\Gamma) = \mathcal{L}_2(\Gamma). \tag{4.6}$$

Proof. It follows from the definition (1.6) of u_Γ and (1.2.A) that $\mathcal{N}_1(\Gamma) \subset \mathcal{L}_1(\Gamma)$ and that

$$u_\Gamma \oplus u_\Gamma = u_\Gamma. \tag{4.7}$$

For every $\nu \in \mathcal{L}_1(\Gamma)$, $u_\Gamma \leqslant u_\Gamma \oplus u_\nu$ and $u_\Gamma \oplus u_\nu \leqslant u_\Gamma \oplus u_\Gamma = u_\Gamma$. Hence, $u_\Gamma \oplus u_\nu = u_\Gamma$, and $\mathcal{L}_1(\Gamma) \subset \mathcal{L}_2(\Gamma)$.

It remains to prove that $\mathcal{L}_2(\Gamma) \subset \mathcal{N}_1(\Gamma)$ if $\Gamma \in \mathbb{F}$. Let μ be the restriction of $\nu \in \mathcal{N}_1$ to Γ^c. Put $u_1 = u_\Gamma \oplus u_\mu$ and $u_2 = u_\Gamma = u_\Gamma \oplus 0$. If $\nu \in \mathcal{L}_2(\Gamma)$, then

$$u_\Gamma \leqslant u_\Gamma \oplus u_\mu \leqslant u_\Gamma \oplus u_\nu = u_\Gamma$$

and therefore $u_1 = u_2$. By (4.3.C), (4.3.D), $\operatorname{tr}^c(u_1) = (\Gamma, \mu)$ and $\operatorname{tr}^c(u_2) = (\Gamma, 0)$. Since $\operatorname{tr}^c(u_1) = \operatorname{tr}^c(u_2)$, we conclude that $\mu = 0$ and $\nu \in \mathcal{N}_1(\Gamma)$. $\square$

Lemma 4.4. *For every $B \in \mathcal{B}$, $\operatorname{tr}^c(u_B) = (S(u_B), 0)$.*

Proof. If $\mathrm{tr}^c(u_B) = (\Gamma, \nu)$, then, by (4.3.C), (4.3.D), $u_B = u_\Gamma \oplus u_\nu$. By (4.7), $u_B \oplus u_B = u_B$. Therefore

$$u_\Gamma \oplus u_\nu = u_B = u_B \oplus u_B = u_\Gamma \oplus u_\nu \oplus u_\Gamma \oplus u_\nu = (u_\Gamma \oplus u_\Gamma) \oplus (u_\nu \oplus u_\nu) = u_\Gamma \oplus u_{2\nu}.$$

Hence, $(\Gamma, \nu) = \mathrm{tr}^c(u_\Gamma \oplus u_\nu) = \mathrm{tr}^c(u_\Gamma \oplus u_{2\nu}) = (\Gamma, 2\nu)$ and $2\nu = \nu$. By (4.2.C), ν vanishes on $R^d = \mathbb{R} \setminus N$ where $N \in \mathcal{P}$ which implies that $u_\nu = 0$ and $u_B = u_\Gamma$. Since $\Gamma \in \mathbb{F}$, the statement of the lemma follows from (4.3.C). □

Proof of Theorem 4.2. 1°. Put $\Gamma = S(u_B)$, $F = \beta(B)$. By Lemma 4.1, $\Gamma \in \mathbb{F}$ and therefore $(\Gamma, 0) \in \mathfrak{C}$. By (4.3.C), $\mathrm{tr}^c(u_\Gamma) = (\Gamma, 0)$ and, by Lemma 4.4, u_B has the same trace. By (4.3.D),

$$u_\Gamma = u_B. \tag{4.8}$$

2°. We claim that

$$B \cap \Gamma^c \sim \emptyset. \tag{4.9}$$

Otherwise there exists a non-null measure $\mu \in \mathcal{N}_1(B \cap \Gamma^c) \subset \mathcal{N}_1(B)$. By (4.5) and (4.8), $\mu \in \mathcal{L}_1(B) = \mathcal{L}_1(\Gamma)$. It follows from Lemma 4.1 that $(\Gamma, \mu) \in \mathfrak{C}$. By (4.3.C),

$$\mathrm{tr}^c(u_\Gamma \oplus u_\mu) = (\Gamma, \mu) \neq (\Gamma, 0) = \mathrm{tr}^c(u_\Gamma)$$

and therefore $\mu \notin \mathcal{L}_2(\Gamma)$. But $\mathcal{L}_1(\Gamma) = \mathcal{L}_2(\Gamma)$ by Lemma 4.3.

3°. Next, we prove that

$$\Gamma \sim F. \tag{4.10}$$

First, we note that, by (4.9), $B = (B \cap \Gamma^c) \cup (B \cap \Gamma) \sim B \cap \Gamma$ and therefore, by (4.2.A),

$$\beta(B) = \beta(B \cap \Gamma) \subset \beta(\Gamma) = \Gamma. \tag{4.11}$$

To prove (4.10), it is sufficient to demonstrate that $\Gamma \cap F^c \sim \emptyset$.

If this is not true, then there exists a measure $\mu \neq 0$ of class $\mathcal{N}_1(\Gamma \cap F^c) \subset \mathcal{N}_1(\Gamma) \subset \mathcal{L}_1(\Gamma)$. Let $\tilde{u} = u_\mu \oplus u_F$.

$$\mathrm{tr}(\tilde{u}) = (F, \mu) \neq (F, 0) = \mathrm{tr}(u_F)$$

and therefore $\tilde{u} \neq u_F$. On the other hand, by (4.11), $u_F \leqslant u_\Gamma$ and, since $u_\mu \leqslant u_\Gamma$, we get $\tilde{u} \leqslant u_\Gamma$. If $u_F = u_\Gamma$, then $u_\Gamma \leqslant u_\Gamma \oplus u_\mu = \tilde{u} \leqslant u_\Gamma$ and therefore $\tilde{u} = u_\Gamma$. But $\mathrm{tr}(\tilde{u}) \neq \mathrm{tr}(u_\Gamma)$ and, by (4.2.C), $\tilde{u} \neq u_\Gamma$.

4°. It follows from (4.10) that $\beta(F) = \beta(\Gamma)$. By Lemma 4.1, $\beta(F) = F$. By (4.2.D), $\Gamma = \beta(\Gamma) \cup Ex(0) = \beta(\Gamma)$. Hence $F = \Gamma$. □

5. Implications of equivalence of DK-trace and MV-trace

Let u be a positive solution of the equation $\Delta u = u^\alpha$ and let $\hat{k}(x, y)$ be the Poisson kernel for the operator $\Delta v - \alpha u^{\alpha-1} v$. It follows from Theorem 4.2 that the set

$$A = \left\{ y \in \partial E\colon\ \hat{k}(x, y) = 0 \text{ for all } x \in E \right\}$$

is equivalent to the set of points y at which Γ is thick, i.e., to the set $\{y\colon\ \Lambda_y(\Gamma) = \infty\}$. ($\Lambda_y(\Gamma)$ is defined by (4.1).) In other words, the symmetric difference of these two sets has the zero Bessel capacity $\mathrm{Cap}_{2/\alpha,\alpha'}$.

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